Symmetry-Induced Modal Characteristics of Uniform Waveguides — I: Summary of Results

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Abstract—The application of symmetry analysis to uniform waveguides is discussed. Symmetry analysis provides exact information concerning mode classification, mode degeneracy, modal electromagnetic-field symmetries, and the minimum waveguide sectors which completely determine the modes in each mode class. Tables are presented which list the possible mode classes and their degeneracies for the two general symmetry families, C_n and C_{nv}, of uniform waveguides. Tables showing the azimuthal dependence of the longitudinal components of the electric and magnetic fields for each mode class are given. Based on this azimuthal dependence, figures showing the minimum waveguide sectors which are necessary and sufficient to completely determine the modes of the various mode classes are presented. The application of symmetry analysis is illustrated by considering uniform waveguides with C_4 and C_{6v} symmetry.

I. INTRODUCTION

IN RECENT YEARS the development of microwave, millimeter, and optical devices and systems has promoted interest in more complex waveguide structures. In order to understand these structures, and to optimize their properties for particular applications, powerful analysis techniques are necessary. As a consequence, high-speed digital computers are now often used for the numerical analysis of such structures. However, the numerical solution of the partial differential equations associated with distributed structures requires extensive computer time if moderate or high accuracy is desired. Therefore, there is a need for analytical techniques which can supplement computer calculations, providing information about the general characteristics of a waveguide, and suggesting possible strategies to minimize the computer time required when particular modes are investigated. In addition, analytical techniques are often preferable to numerical calculations when a general understanding of the propagation characteristics of a waveguide is sought. Symmetry analysis is one analytical technique which can provide basic information concerning the modal characteristics and suggest possible strategies to optimize computer studies of particular structures.

The symmetry of a waveguide controls several of the important characteristics of the modes of the waveguide. A determination of the symmetry type of a particular waveguide enables one to classify the possible modes into mode classes, predict the mode degeneracies between mode classes, and determine the azimuthal symmetries of the modal electromagnetic fields in each mode class. Further, one can specify minimum waveguide sectors for each mode class which completely determine the modes of that mode class. All of this can be accomplished from a knowledge of the waveguide symmetry without having to solve the boundary value problem for the particular waveguide structure.

In this paper, attention is restricted to uniform waveguides which may be transversely inhomogeneous, but whose media are isotropic and piecewise homogeneous. This restricted class of waveguides includes most structures of current interest, except for those waveguides containing gyrotropic media such as ferrites (uniform waveguides with gyrotropic media will be discussed in a future paper). This restriction enables us to provide tables of the mode classes, mode degeneracies, azimuthal modal field symmetries, and minimum waveguide sectors for any waveguide of this type. These waveguides may be lossy or lossless, and have either a closed or open boundary.

This discussion of the symmetry-induced modal characteristics of uniform waveguides is presented in two parts: "I: Summary of Results" and "II: Theory." The symmetry analysis of waveguides is based on group theory, and, in particular, on the theory of group representations. However, at least for the waveguides in the restricted class considered here, it is not necessary to have a knowledge of group theory in order to apply the results of symmetry analysis to specific waveguides of interest. It is only necessary to be able to identify the symmetry operations belonging to the structure under study. In order to make these results of symmetry analysis as widely accessible to microwave engineers as possible, the "Summary of Results" is presented first, and no group theoretical development is included in this paper. For those interested in how these results are obtained, the theory leading to them is discussed briefly in the accompanying paper, "II: Theory."

Throughout this paper it is assumed that the waveguides under discussion are inhomogeneous. Therefore, the waveguide modes are, in general, hybrid modes with longitudinal components of both the electric and magnetic fields. Homogeneous waveguides are a special case, and the results listed here apply to them with some obvious simplifications in the modal field representations.

It is well known that the transverse electric and magnetic fields in a uniform waveguide can be expressed in terms of the longitudinal components. For simplicity, only the longitudinal components of the electric and mag-

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netic fields will be included in the discussion of the azimuthal symmetry of the modal fields. For any particular structure the azimuthal symmetry of the transverse components of the fields can be readily inferred from the azimuthal symmetry of the longitudinal components.

II. SYMMETRY OF A UNIFORM WAVEGUIDE

Taking the waveguide axis as the z axis, a uniform waveguide of infinite length is invariant to translations parallel to the z axis. For an exp $(j\omega t)$ time dependence, this leads to a set of modes which vary as exp $(-\gamma z)$. Here γ is the propagation constant which is a function of ω and, in general, a complex number. A waveguide with a closed boundary (opaque to electromagnetic fields) has a discrete mode spectrum with an infinite number of discrete values of γ for each ω. If no opaque transverse boundary is present, the waveguide is said to have an open boundary, and for a given value of ω the mode spectrum consists of a finite number of discrete modes plus a continuous spectrum. The discussion to follow applies to waveguides with either type of boundary.

The solution for the modal electromagnetic fields of a waveguide at a particular frequency entails solving an eigenvalue problem where the eigenvalues are the values of $\gamma(\omega)$. For any mode of a uniform waveguide, the transverse electric and magnetic fields can be expressed in terms of the longitudinal components, E_z and H_z [1]. Therefore, pairs of E_z and H_z form the eigenfunctions for the problem. For a uniform waveguide the partial differential equations and boundary conditions for E_z and H_z involve only the transverse coordinates (see the following paper, Section II; hereafter, such references will be given as [II-II]). As a consequence, only the symmetry of the waveguide cross section need be considered. This restricts the relevant waveguide symmetry types to just two general families.

A symmetry operation for a figure is a spatial operation which leaves the figure unchanged in appearance. For a two-dimensional figure, only two types of spatial symmetry operations can exist; rotations about a symmetry axis oriented normal to the plane of the figure, and reflections in planes oriented normal to the plane of the figure.

In general, for a plane figure, if the smallest angle of rotation which causes the pattern to appear unchanged is $2\pi/n$ rad, then all the possible inequivalent rotational symmetry operations of the figure are included in the set of n operations: $C_n, C_n^2, C_n^3, \cdots, C_n^{n-1}, C_n^n = E$. Here C_n denotes rotation by $2\pi/n$ rad and E denotes the identity operation. A pattern which possesses only rotational symmetry (no reflection symmetry), and for which $2\pi/n$ is the smallest angle associated with a symmetry operation, is said to possess the symmetry group C_n of order n. The symbol C_n stands for both a particular symmetry operation and the collection of all symmetry operations based on it. Fig. 1 shows the cross sections of several waveguides with C_n symmetry.

A plane figure may also possess reflection symmetries. If a plane figure has n-fold rotation symmetry and also possesses at least one plane of reflection symmetry, then there are precisely n planes of reflection symmetry. These planes all intersect along the axis of rotational symmetry and are spaced azimuthally at π/n rad. The total number of symmetry operations is n rotations plus n reflections, or $2n$ symmetry operations. The symmetry group for such a figure is designated as C_{nv} (of order $2n$). Fig. 2 shows the cross sections of several waveguides with C_{nv} symmetry.

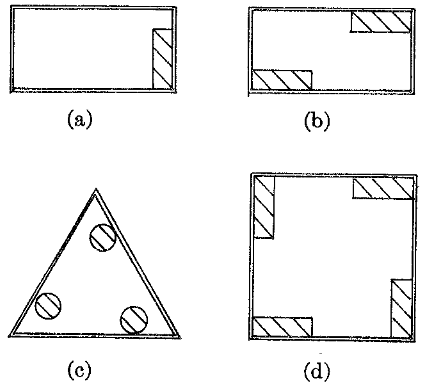

Fig. 1. Uniform waveguides with C_n symmetry. (a) C_1. (b) C_2. (c) C_3. (d) C_4.

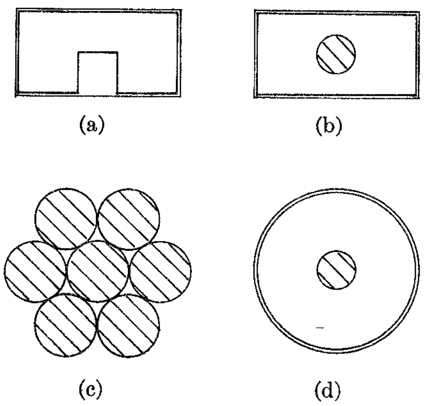

Fig. 2. Uniform waveguides with C_{nv} symmetry. (a) C_{1v}. (b) C_{2v}. (c) C_{6v}. (d) $C_{\infty v}$.

These two families of symmetry groups, C_n and C_{nv}, exhaust the possibilities for uniform inhomogeneous waveguides with isotropic media. Note that n may be any integer from one to infinity. Fig. 2(d) shows an example of a waveguide with $C_{\infty v}$ symmetry. An example of a uniform waveguide with C_∞ symmetry is a sheath helix [2]. A sheath helix must be either right- or left-handed, and a reflection transforms one helix type into the other. Therefore, reflection is not a symmetry operation for a sheath helix.

III. MODE CLASSIFICATION AND DEGENERACY

Every uniform waveguide has a mode spectrum containing an infinite number of modes. However, the number of distinct azimuthal symmetries of the modal electromagnetic-field patterns for a structure of a given symmetry type, C_n or C_{nv}, is of the order of n [II–IV]. Thus the modes of a waveguide can be assigned to classes depending on the azimuthal symmetry of the modal field patterns. These classes will be called mode classes, and each contains an infinite number of modes.

The electromagnetic fields for all the modes in a given mode class will have the same azimuthal symmetry, although the detailed dependence of the electromagnetic fields on the aximuthal coordinate will differ. The major differences between the electromagnetic-field patterns for different modes in the same mode class lie in their radial variations. For waveguides with C_n symmetry there are n distinct mode classes, while for waveguides with C_{nv} symmetry there are either $n+1$ (for n odd) or $n+2$ (for n even) distinct mode classes [II–IV].

The electromagnetic fields of a particular mode can be labeled by a double subscript; for example, $\boldsymbol{E}_{pq}$ and $\boldsymbol{H}_{pq}$. The first subscript, p, designates the mode class, while the second subscript, q, distinguishes the modes within the pth mode class. For waveguides with C_n symmetry, $1 \leq p \leq n$, while for waveguides with C_{nv} symmetry, $1 \leq p \leq n+1$ if n is odd, or $1 \leq p \leq n+2$ if n is even. In all cases of closed-boundary waveguides, q will be a positive integer in the range from one to infinity. For the discrete modes of an open-boundary waveguide, q will be a positive integer, but usually limited to only a few possible values. For the continuous spectrum the mode class label p is still valid, but the second subscript q may not be useful.

It can be shown [II–IV] that either all the modes in a mode class are nondegenerate, or that all the modes in a mode class are degenerate with the corresponding modes in a complementary mode class. Mode degeneracies can occur only in pairs. For a given waveguide symmetry group, the number of nondegenerate and degenerate mode classes is known. Tables I and II present the mode classes and their degeneracies for waveguides with C_n and C_{nv} symmetry, respectively.

A comment about the possible degeneracies of a uniform waveguide should be made. The degeneracies cataloged here are those produced by the waveguide symmetry and will occur for all values of ω. For some inhomogeneous waveguides, however, the curves of $\gamma(\omega)$ versus ω for two or more different modes may happen to cross at a particular value of ω, producing a degeneracy at a discrete frequency which is not related to the symmetry of the waveguide. This is termed an "accidental" degeneracy, and symmetry analysis cannot predict such isolated degeneracies.

At this point an alert reader may be suspicious of the validity of Table II, because it apparently incorrectly predicts the mode degeneracies of the most commonly analyzed waveguides; homogeneous rectangular, square, and circular waveguides with perfectly conducting walls. However, symmetry analysis can correctly predict the mode degeneracies of a waveguide only if *all* the symmetry operations of the structure are accounted for. Not all symmetry operations involve spatial rotations and reflections. These three waveguides are special in that they include an additional "hidden" symmetry which increases the number of mode degeneracies. This "hidden" symmetry will be discussed in Section VI, together with a nonspatial symmetry common to all the waveguides considered in this paper.

TABLE I
TABLE OF MODE CLASSES AND MODE DEGENERACIES FOR UNIFORM WAVEGUIDES WITH C_n SYMMETRY

n	Number of non-degenerate mode classes	Number of pairs of two-fold degenerate mode classes	Total number of mode classes
odd	1	(n - 1)/2	n
even	2	(n - 2)/2	n
∞	1	∞	∞

TABLE II
TABLE OF MODE CLASSES AND MODE DEGENERACIES FOR UNIFORM WAVEGUIDES WITH C_{nv} SYMMETRY

n	Number of non-degenerate mode classes	Number of pairs of two-fold degenerate mode classes	Total number of mode classes
odd	2	(n - 1)/2	n + 1
even	4	(n - 2)/2	n + 2
∞	2	∞	∞

Here it is sufficient to state that Tables I and II apply to all inhomogeneous waveguides of the general class considered, and to all homogeneous waveguides with the exception of the three special cases of homogeneous rectangular, square, and circular waveguides with closed boundaries. For example, Table II correctly predicts the mode characteristics of a homogeneous waveguide of elliptical cross section with a perfectly conducting wall, whose modes have been tabulated [3]. The elliptical waveguide has the same symmetry group, C_{2v}, as a rectangular waveguide.

IV. MODAL ELECTROMAGNETIC-FIELD SYMMETRIES

The characteristic that physically distinguishes the mode classes of a particular waveguide is the azimuthal symmetry of the electromagnetic fields. One way to display analytically the azimuthal symmetry of the longitudinal components of the electric and magnetic fields is to express them in terms of Fourier series in the azimuthal angle θ. For example, it is found that the longitudinal electric- and magnetic-field components for the modes in the only nondegenerate mode class of a waveguide with C_3 symmetry [such as that shown in Fig. 1(c)] can be written as

$$E_{z1q}(r,\theta) = \sum_{\nu=-\infty}^{\infty} A_{1q\nu}(r) \exp(j3\nu\theta)$$

$$H_{z1q}(r,\theta) = \sum_{\nu=-\infty}^{\infty} B_{1q\nu}(r) \exp(j3\nu\theta). \tag{1}$$

Here, the subscripts indicate that this is the qth mode of

TABLE III

FOURIER SERIES REPRESENTATIONS OF THE LONGITUDINAL ELECTRIC AND MAGNETIC FIELDS FOR UNIFORM WAVEGUIDES WITH C_n SYMMETRY

n	Mode class p	E_{zpq}	H_{zpq}
even, odd	1	$\sum_{\nu=-\infty}^{\infty} A_{1q\nu}(r)\, e^{jn\nu\theta}$	$\sum_{\nu=-\infty}^{\infty} B_{1q\nu}(r)\, e^{jn\nu\theta}$
even, odd	k	$\sum_{\nu=-\infty}^{\infty} A_{kq\nu}(r)\, e^{j(n\nu+k/2)\theta}$	$\sum_{\nu\ \infty}^{\infty} B_{kq\nu}(r)\, e^{j(n\nu+k/2)\theta}$
even, odd	k+1	$\sum_{\nu=-\infty}^{\infty} A_{(k+1)q\nu}(r)\, e^{j(n\nu-k/2)\theta}$	$\sum_{\nu=-\infty}^{\infty} B_{(k+1)q\nu}(r)\, e^{j(n\nu-k/2)\theta}$
even	n	$\sum_{\nu=-\infty}^{\infty} A_{nq\nu}(r)\, e^{jn(\nu-1/2)\theta}$	$\sum_{\nu=-\infty}^{\infty} B_{nq\nu}(r)\, e^{jn(\nu-1/2)\theta}$
∞	1	$A_{1q}(r)$	$B_{1q}(r)$
∞	k	$A_{kq}(r)\, e^{jk\theta/2}$	$B_{kq}(r)\, e^{jk\theta/2}$
∞	k+1	$A_{(k+1)q}(r)\, e^{-jk\theta/2}$	$B_{(k+1)}(r)\, e^{-jk\theta/2}$

Note: Mode classes $k,k+1$ are a degenerate pair; k is even.

the first mode class. Note that in the Fourier series, only one-third of the possible terms that would occur in a general Fourier series are present for this mode class. This result for this mode class is a consequence of the symmetry of the waveguide [II–IV]. Symmetry analysis gives no information, however, about the magnitudes of the coefficients $A_{1q\nu}(r)$ and $B_{1q\nu}(r)$. To determine these coefficients one must solve the partial differential equations for the system subject to the appropriate boundary conditions. One can conclude, however, from the Fourier series in (1) that the electromagnetic fields for modes in this mode class must be periodic in θ with period $2\pi/3$ rad. Thus, in any numerical analysis of the nondegenerate modes of this waveguide, all of which belong to this mode class (see Table I), only a sector of angle $2\pi/3$ rad need be considered (with the boundary conditions that the electromagnetic fields must be identical at the two azimuthal boundaries of the sector).

Although the azimuthal symmetry of the longitudinal electric and magnetic fields for the various mode classes will be presented by writing these field components in terms of Fourier series, it is *not* suggested that this is a preferred form for making a detailed numerical analysis. Other representations may well be preferable for a computer study of a particular waveguide. The purpose in using the Fourier series representation here is to be able to extract information easily concerning the azimuthal symmetry of the modal electromagnetic fields.

Table III presents the general form of the Fourier series for the longitudinal components of the electric and magnetic fields for waveguides with C_n symmetry. Referring to Table I, waveguides with C_n symmetry will have either one (n odd) or two (n even) mode classes containing nondegenerate modes. It is convenient to label the mode class containing the nondegenerate modes which occurs for n either even or odd as the first mode class ($p = 1$), and the mode class containing nondegenerate modes which occurs only for n even as the last mode class ($p = n$). The mode-class pairs which combine to give the twofold degenerate modes are listed from $p = 2$ to $p = n$ (n odd), or to $p = n - 1$ (n even). Thus mode classes $p = 2$ and 3, 4, and 5, etc., are pairs with mutually degenerate modes. The results for the limiting case C_∞ are also given; only a single mode class with nondegenerate modes occurs in this case. Note that Table III gives the explicit azimuthal dependence of E_z and H_z for the mode classes of waveguides with C_∞ symmetry.

Table IV presents the general form of the Fourier series for the longitudinal components of the electric and magnetic fields for waveguides with C_{nv} symmetry. Referring to Table II, waveguides with C_{nv} symmetry have either two (n odd) or four (n even) mode classes containing nondegenerate modes. It is convenient to label the two mode classes containing nondegenerate modes which occur for n either even or odd as the first and second mode classes ($p = 1,2$). The two mode classes containing nondegenerate modes which occur only for n even are placed at the end of the list ($p = n + 1, n + 2$). The mode-class pairs which combine to give two fold degenerate modes are listed from $p = 3$ to $p = n + 1$ (n odd), or to $p = n$ (n even). The results for the limiting case $C_{\infty v}$ are also given; only two mode classes with nondegenerate modes occur in this case. Again Table IV gives the explicit azimuthal dependence for the mode classes of waveguides with $C_{\infty v}$ symmetry. It is important to note that the Fourier series as written in Table IV assume that $\theta = 0$ is chosen to coincide with one of the planes of reflection symmetry of the structure under consideration.

V. MINIMUM WAVEGUIDE SECTORS

For a given waveguide, the information presented in Tables III and IV enables one to specify for each mode class of the waveguide a miniumum sector of the waveguide cross section which is sufficient, and necessary, to com-

TABLE IV
FOURIER SERIES REPRESENTATIONS OF THE LONGITUDINAL ELECTRIC AND MAGNETIC FIELDS FOR UNIFORM WAVEGUIDES WITH C_{nv} SYMMETRY

n	Mode class p	E_{zpq}	H_{zpq}
even, odd	1	$\sum_{\nu=0}^{\infty} A_{1q\nu}(r)\cos(n\nu\theta)$	$\sum_{\nu=0}^{\infty} B_{1q\nu}(r)\sin(n\nu\theta)$
even, odd	2	$\sum_{\nu=0}^{\infty} A_{2q\nu}(r)\sin(n\nu\theta)$	$\sum_{\nu=0}^{\infty} B_{2q\nu}(r)\cos(n\nu\theta)$
even, odd	k	$\sum_{\nu=0}^{\infty} (A_{kq\nu}(r)\cos[(n\nu-(k-1)/2)\theta] + C_{kq\nu}(r)\cos[(n\nu+(k-1)/2)\theta])$	$\sum_{\nu=0}^{\infty} (B_{kq\nu}(r)\sin[(n\nu-(k-1)/2)\theta] + D_{kq\nu}(r)\sin[(n\nu+(k-1)/2)\theta])$
even, odd	k+1	$\sum_{\nu=0}^{\infty} (A_{(k+1)q\nu}(r)\sin[(n\nu-(k-1)/2)\theta] + C_{(k+1)q\nu}(r)\sin[(n\nu+(k-1)/2)\theta])$	$\sum_{\nu=0}^{\infty} (B_{(k+1)q\nu}(r)\cos[(n\nu-(k-1)/2)\theta] + D_{(k+1)q\nu}(r)\cos[(n\nu+(k-1)/2)\theta])$
even	n+1	$\sum_{\nu=0}^{\infty} A_{(n+1)q\nu}(r)\cos[n(\nu+1/2)\theta]$	$\sum_{\nu=0}^{\infty} B_{(n+1)q\nu}(r)\sin[n(\nu+1/2)\theta]$
even	n+2	$\sum_{\nu=0}^{\infty} A_{(n+2)q\nu}(r)\sin[n(\nu+1/2)\theta]$	$\sum_{\nu=0}^{\infty} B_{(n+2)q\nu}(r)\cos[n(\nu+1/2)\theta]$
∞	1	$A_{1q}(r)$	0
∞	2	0	$B_{2q}(r)$
∞	k	$A_{kq}(r)\cos[(k-1)\theta/2]$	$B_{kq}(r)\sin[(k-1)\theta/2]$
∞	k+1	$A_{(k+1)q}(r)\sin[(k-1)\theta/2]$	$B_{(k+1)q}(r)\cos[(k-1)\theta/2]$

Note: Mode classes $k,k+1$ are a degenerate pair; k is odd.

pletely determine the modal eigenvalues and electromagnetic fields for all of the modes of that mode class. Figs. 3 and 4 show the minimum sectors for the mode classes of waveguides with C_n and C_{nv} symmetry, respectively. In each case, the minimum subregion of the waveguide cross section is a sector with the vertex of the sector angle located at the waveguide axis. These figures give the magnitude in radians of the azimuthal angle of the minimum sector and specify the boundary conditions for the electromagnetic fields on the two straight lines bounding the sector. It is not necessary to present figures for the two limiting cases of waveguides with C_∞ or $C_{\infty v}$ symmetry, because Tables III and IV give the explicit azimuthal dependence of the longitudinal components of the electric and magnetic fields for the various mode classes in these cases.

In Fig. 3 the boundary lines of the minimum waveguide sectors are shown either as dotted lines, or as dot–dash lines. Dotted lines indicate periodic boundary conditions; that is, the electromagnetic fields on the two dotted lines must be identical. Dot-dash lines indicate "quasi-periodic" boundary conditions; that is, the electromagnetic fields on these two lines are identical except that the sign of the fields along one line is reversed relative to the fields along the other line. In Fig. 4 the boundary lines are shown either solid or dashed. Solid lines indicate a short-circuit boundary condition (tangential electric field is zero), and dashed lines indicate an open-circuit boundary condition (tangential magnetic field is zero). For waveguides with C_n symmetry there is no particular relationship between the boundary lines of the sectors shown in Fig. 3 and a physical characteristic of the waveguide; that is, any sector of the specified angle could be used. In the case of waveguides with C_{nv} symmetry, however, the boundary lines of the minimum waveguide sectors shown in Fig. 4 must always coincide with two of the planes of reflection symmetry of the waveguide structure.

In determining the minimum waveguide sectors of the degenerate mode class pairs one finds that there are a number of special cases possible, particularly as n increases. Therefore, in order to facilitate the use of Figs. 3 and 4, Tables V and VI are presented. To use these tables together with Figs. 3 and 4 for a particular waveguide, three steps should be followed.

1) Determine the symmetry type of the waveguide.

2) Determine the number of nondegenerate and degenerate mode classes (see Tables I and II).

3) For the particular mode class of interest enter Table V (for C_n symmetry) or Table VI (for C_{nv} symmetry) at the appropriate row. The various columns give the minimum sector angles, the boundary conditions, and refer to the relevant portions of Figs. 3 or 4.

There are several different cases possible for degenerate mode-class pairs for waveguides with C_n symmetry. In each of these, however, the minimum waveguide sector angle and boundary conditions are the same for both

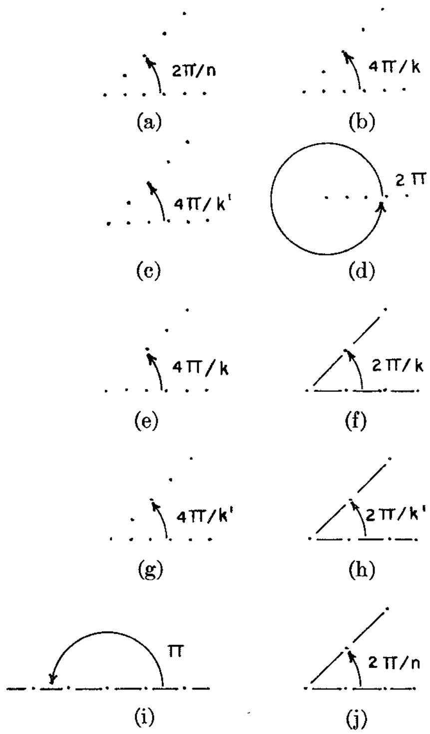

Fig. 3. Minimum sectors for waveguides with C_n symmetry (refer to Table V). (a) Nondegenerate mode class, $p = 1$. (b)–(i) Degenerate mode classes, $p = k,\ k + 1$. (j) Nondegenerate mode class, $p = n$.

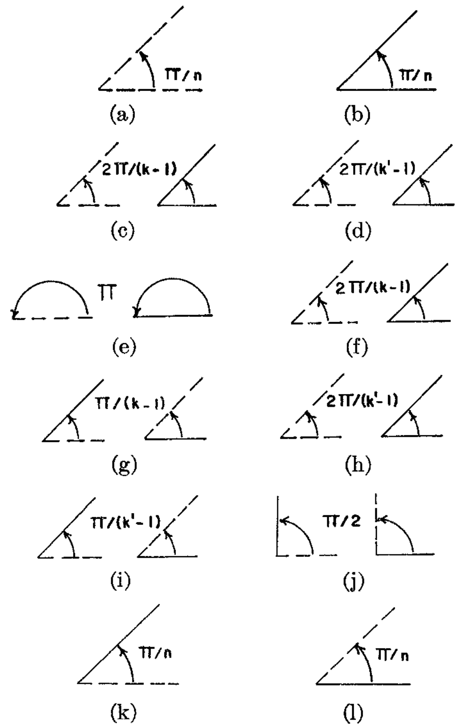

Fig. 4. Minimum sectors for waveguides with C_{nv} symmetry (refer to Table VI). (a) Nondegenerate mode class, $p = 1$. (b) Nondegenerate mode class, $p = 2$. (c)–(j) Degenerate mode classes, $p = k,\ k + 1$. (k) Nondegenerate mode class, $p = n + 1$. (l) Nondegenerate mode class, $p = n + 2$.

TABLE V
MINIMUM SECTORS FOR WAVEGUIDES WITH C_n SYMMETRY

n	Mode class p	Degenerate	If Degenerate: $\frac{k}{2} = \frac{n}{m}$	If Degenerate: $k = uk'$, $\frac{k'}{2} = \frac{n}{m}$	Minimum sector angle, radians	Boundary conditions	Figure
even, odd	1	No			$2\pi/n$	periodic	3a
odd	k, k+1	Yes	Yes		$4\pi/k$	periodic	3b
odd	k, k+1	Yes	No	Yes	$4\pi/k'$	periodic	3c
odd	k, k+1	Yes	No	No	2π	periodic	3d
even	k, k+1	Yes	Yes; m odd		$4\pi/k$	periodic	3e
even	k, k+1	Yes	Yes; m even		$2\pi/k$	quasi-periodic	3f
even	k, k+1	Yes	No	Yes; m odd	$4\pi/k'$	periodic	3g
even	k, k+1	Yes	No	Yes; m even	$2\pi/k'$	quasi-periodic	3h
even	k, k+1	Yes	No	No	π	quasi-periodic	3i
even	n	No			$2\pi/n$	quasi-periodic	3j

mode classes of the pair k and $k + 1$. The cases are distinguished by whether $k/2$ is an integer divisor of n ($k/2 = n/m$, with $m < n$), an integer multiple of an integer divisor of n ($k = uk'$, $k'/2 = n/m$, with $m < n$), or neither. The various possibilities, and their consequences, are displayed in Table V and Fig. 3. In those cases where $k = uk'$ with $k'/2 = n/m$, and there is a choice of several possible values of k', one should always select the largest of the possible values of k' (smallest possible value of m).

There are also several different cases possible for degenerate mode-class pairs for waveguides with C_{nv} symmetry. The cases are distinguished by whether $(k - 1)/2$ is an integer divisor of n ($(k - 1)/2 = n/m$, with $m < n$), an integer multiple of an integer divisor of n ($(k - 1) = u(k' - 1)$, $(k' - 1)/2 = n/m$, with $m < n$), or neither. The various possibilities, and their consequences, are displayed in Table VI and Fig. 4. In those cases where $(k - 1) = u\,k' - 1)$ with $(k' - 1)/2 = n/m$, and there

TABLE VI
MINIMUM SECTORS FOR WAVEGUIDES WITH C_{nv} SYMMETRY

n	Mode class p	Degenerate	If Degenerate $\frac{k-1}{2} = \frac{n}{m}$	If Degenerate $k-1 = u(k'-1)$, $\frac{k'-1}{2} = \frac{n}{m}$	Minimum sector angle, radians	Boundary conditions	Figure
even, odd	1	No			π/n	open circuit	4a
even, odd	2	No			π/n	short circuit	4b
odd	k, k+1	Yes	Yes		$2\pi/(k-1)$	open circuit short circuit	4c
odd	k, k+1	Yes	No	Yes	$2\pi/(k'-1)$	open circuit short circuit	4d
odd	k, k+1	Yes	No	No	π	open circuit short circuit	4e
even	k, k+1	Yes	Yes; m odd		$2\pi/(k-1)$	open circuit short circuit	4f
even	k, k+1	Yes	Yes; m even		$\pi/(k-1)$	short and open circuit	4g
even	k, k+1	Yes	No	Yes; m odd	$2\pi/(k'-1)$	open circuit short circuit	4h
even	k, k+1	Yes	No	Yes; m even	$\pi/(k'-1)$	short and open circuit	4i
even	k, k+1	Yes	No	No	$\pi/2$	short and open circuit	4j
even	n+1	No			π/n	short and open circuit	4k
even	n+2	No			π/n	short and open circuit	4ℓ

is a choice of several possible values of k', one should always select the largest of the possible values of k' (smallest possible value of m).

Note that while the minimum waveguide-sector angle is the same for both mode classes of a degenerate pair, the boundary conditions are different for each mode class of the pair. This result for the degenerate mode classes of waveguides with C_{nv} symmetry contrasts with that for waveguides with C_n symmetry, where both mode classes of a degenerate pair have the same sector boundary conditions. Thus, in the case of waveguides with C_{nv} symmetry, the use of the minimum waveguide sector for analyzing the modes in a degenerate mode class leads to an important benefit in addition to minimizing the waveguide area that is necessary to be included in the analysis. By using the minimum waveguide sector with its appropriate boundary conditions, the degeneracy of the modes is lifted; therefore, in any numerical calculations each mode can be treated as a nondegenerate mode.

Two examples will be examined briefly to illustrate the application of symmetry analysis to specific structures. First, consider the hollow conducting pipe of square cross section with four dielectric slabs located so as to produce a structure with C_4 symmetry, Fig. 5(a). From Table I, this waveguide has a total of four mode classes. Mode classes 1 and 4 have nondegenerate modes, while mode classes 2 and 3 form a pair with mutually degenerate modes. Using Table V and Fig. 3, one finds that the minimum waveguide sectors which are necessary and sufficient to determine the modal electromagnetic fields are those shown in Fig. 5. Fig. 5(b) and 5(d) show the minimum sectors for mode classes 1 and 4, respectively, while Fig. 5(c) shows the minimum sector for the pair of degenerate mode classes (2 and 3). The particular sectors shown are not unique; other sectors with the same angle and boundary conditions could have been chosen instead.

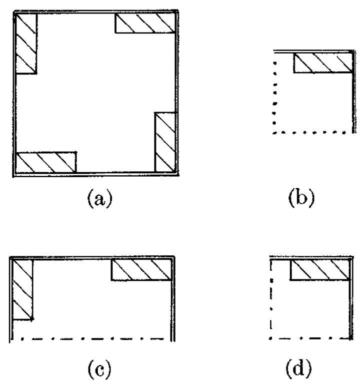

Fig. 5. Minimum waveguide sectors for a waveguide with C_4 symmetry. (a) Waveguide with C_4 symmetry. (b) First mode class (nondegenerate). (c) Second and third mode classes (degenerate pair). (d) Fourth mode class (nondegenerate). Dotted lines indicate periodic boundary conditions; dot–dash lines indicate quasi-periodic boundary conditions.

The open-boundary waveguide of Fig. 6(a) has seven dielectric rods (or fibers) of equal diameter arranged in a close-packed structure with C_{6v} symmetry. From Table II, this waveguide has a total of eight mode classes. Mode classes 1, 2, 7, and 8 are nondegenerate, while mode classes 3 and 4, 5 and 6, are two pairs, with each pair having mutually degenerate modes. Using Table VI and Fig. 4, one finds that the minimum sectors which are necessary and sufficient to determine the modal electromagnetic fields of these mode classes are those shown in Fig. 6. Fig. 6(b), (c), (f), and (g) show the minimum sec-

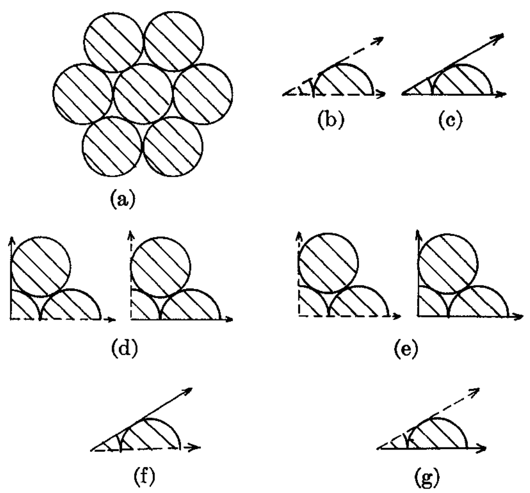

Fig. 6. Minimum waveguide sectors for a waveguide with C_{6v} symmetry. (a) Waveguide with C_{6v} symmetry. (b) First mode class (nondegenerate). (c) Second mode class (nondegenerate). (d) Third and fourth mode classes (degenerate pair). (e) Fifth and sixth mode classes (degenerate pair). (f) Seventh mode class (nondegenerate). (g) Eighth mode class (nondegenerate). Solid lines indicate short-circuit boundary conditions; dashed lines indicate open-circuit boundary conditions.

tors for mode classes 1, 2, 7, and 8, respectively. Fig. 6(d) shows the minimum sectors for the mode class pair 3 and 4, and Fig. 6(e) for the mode class pair 5 and 6.

VI. NONSPATIAL SYMMETRY

In Section III there was a brief reference to nonspatial symmetries which may influence the modal characteristics of a waveguide. An important example of such a nonspatial symmetry for waveguides is "frequency-reversal" symmetry [II–V]. This symmetry is a consequence of the requirement that $\epsilon^*(-\omega) = \epsilon(\omega)$ and $\mu^*(-\omega) = \mu(\omega)$ for real ω. This additional symmetry operation has no effect on the modal characteristics of waveguides with C_{nv} symmetry. However, this symmetry operation does play an important role in the modal characteristics of waveguides with C_n symmetry. Without the inclusion of this symmetry operation, all of the mode classes of waveguides with C_n symmetry would be nondegenerate; the occurrence of degenerate pairs of mode classes for these waveguides depends on the presence of the frequency-reversal symmetry operation. All of the results given for waveguides with C_n symmetry in the previous sections include its influence.

A second example of a nonspatial symmetry is one which occurs only for homogeneous waveguides with perfectly conducting boundaries which are either square, rectangular, or circular (see Section III). In this case the additional symmetry depends on the special geometry of these waveguides (note that homogeneous waveguides with rectangular and elliptical boundaries belong to the same symmetry group, C_{2v}, but their geometries are different). For homogeneous waveguides with square or rectangular walls, the transverse dependence of the axial electric field for the E modes can be written as the product of two trigonometric functions; for example,

$$E_z(x,y) = A_{mn} \sin(m\pi x/a) \sin(n\pi y/b)$$

with $m,n \geq 1$, if the waveguide width and height are a and b, respectively. By applying the differential operator $\partial/\partial x\, \partial/\partial y$ to this field, one obtains a function $B_{mn} \cos(m\pi x/a) \cos(n\pi y/b)$, which is characteristic of the axial magnetic field of the H modes of the waveguide. This operator reflects a geometric symmetry of these waveguides which produces degeneracies between the E and H modes in addition to those tabulated in Table II. A similar geometry-induced symmetry occurs for homogeneous waveguides with perfectly conducting circular walls. In this case, additional degeneracies are produced between the E_{1n} and H_{0n} modes; this result applies to hollow circular waveguides and coaxial circular waveguides.

It is believed that these three cases of homogeneous waveguides with square, rectangular, or circular walls which are perfectly conducting, are the only ones of practical interest which show additional geometry-induced mode degeneracies. Homogeneous waveguides with perfectly conducting walls whose cross sections are other than square, rectangular, or circular will not show geometry-induced mode degeneracies, nor will any inhomogeneous waveguide, regardless of the boundary geometry. Since the modal characteristics of homogeneous waveguides with square, rectangular, and circular walls are well established and discussed in many textbooks, there is no point in applying the symmetry analysis described in this paper to such waveguides. Therefore, their exclusion here is not a significant restriction on this symmetry analysis.

VII. DISCUSSION

Symmetry analysis provides exact information concerning the following characteristics of the modes of uniform waveguides: the classification of the modes into mode classes; the possible degeneracies of the modes; the azimuthal symmetries of the modal electromagnetic fields for each mode class; and the minimum waveguide sectors which are necessary and sufficient to completely determine the modes in each mode class. The results obtained here are applicable to waveguides which may be transversely inhomogeneous, but whose media are isotropic and piecewise homogeneous. The waveguide may be lossy or lossless and have either an open or closed boundary. Because all of the uniform waveguides considered in this paper are included in the two general symmetry families, C_n and C_{nv}, it has been possible to tabulate the results for all possible cases.

It should be clear that symmetry analysis cannot provide complete information concerning all of the modal characteristics of uniform waveguides. For example, it can provide no direct information concerning the ordering of the waveguide modes based on the cutoff frequencies. In addition, the results are exact. That means, for example, that symmetry analysis states that modes are either nondegenerate or are degenerate. It cannot indicate when modes are "almost" degenerate.

In order to exploit symmetry analysis fully, one must use "common sense" in applying it to particular structures. For example, suppose the waveguide under consideration has a particular symmetry type, but its cross section is such that it "almost" has a higher symmetry type. This waveguide may well have mode classes which are nearly degenerate, and one would be advised to study the implications of both symmetry types to predict the modal characteristics the structure would exhibit. Actually, a deeper exploration of symmetry analysis can indicate how the degeneracies of modes are split when the symmetry is "lowered;" this would require some knowledge of group representation theory and is not considered here.

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Symmetry-Induced Modal Characteristics of Uniform Waveguides — II: Theory

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Abstract—The application of symmetry analysis to uniform waveguides is discussed. Symmetry analysis provides exact information concerning mode classification, mode degeneracy, modal electromagnetic-field symmetries, and the minimum waveguide sectors which completely determine the modes in each mode class. This paper provides a summary of the development that leads to the results concerning symmetry-induced modal characteristics of uniform waveguides discussed in the previous paper. Some of the concepts of group theory are introduced, including the irreducible representations of symmetry groups. The use of the irreducible representations to determine the mode classes and their degeneracies is described. The projection operators belonging to the irreducible representations are introduced and their application to determining the azimuthal symmetry of the modal fields is explained. The minimum waveguide sectors for the mode classes are obtained from the azimuthal symmetry of the modal fields.

I. INTRODUCTION

THE PURPOSE of this paper is to provide a summary of the development that leads to the results concerning the symmetry-induced modal characteristics of uniform waveguides discussed in the previous paper. These results are based on group theory and, in particular, on the theory of group representations. There have been many applications of group theory to various branches of physics and chemistry, and the literature describing these applications is copious. However, there have been few applications of group theory to the field of microwaves. One exception is symmetrical waveguide junctions which have been investigated by Montgomery *et al.* [1], Kerns [2], and Auld [3]. A few papers have been published which explored the consequences of symmetry in periodic waveguides. Two recent publications are [4] and [5]; the second paper employs group-theoretic methods. There has been little attention given, however, to exploiting the role symmetry plays in determining the modal characteristics of uniform waveguides.

A coherent exposition of the development of the complete theory required for the symmetry analysis of uniform waveguides starting from the basic concepts of group theory is not feasible in the few pages appropriate to a journal paper, and this is not attempted here. Instead, the relevant results from group theory will be cited, and a brief indication given how these lead to the results presented for uniform waveguides in the previous paper (hereafter referred to as [I]). This paper is not intended to enable a reader unfamiliar with group theory to attain a working knowledge of it as a technique for application to microwave analysis. However, it is hoped that these papers may provide a glimpse of the power of this technique and motivate some readers to explore it. Three of the many excellent books on the application of group theory to various branches of physics and chemistry are [6]–[8]. To provide the maximum assistance to any interested

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